Interface roughening in systems with quenched disorder

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We present a theoretical framework for the discussion of the scaling properties of interfaces advancing in systems with quenched disorder. In all such systems there are critical conditions at which the interface gains scale invariance for sufficiently slow growth. There are two fundamental concepts, the "blocking surfaces" and the "associated processes," whose nature determines the scaling properties of the advancing interfaces at criticality. The associated processes define a network whose scaling properties determine all the exponents (static and dynamic) that characterize the critical growing interface via universal scaling relations. We point out in this paper that most of the physical rules that can be used to advance the interface also incorporate noncritical elements; as a result, the roughness exponent of the growing interface may deviate from that of the critical interface in a rule-dependent way. We illustrate the wide applicability of the universal scaling relations with diverse models, such as the Edwards-Wilkinson (EW) model with quenched noise, the random-field Ising model, and the Kardar-Parisi-Zhang (KPZ) model with quenched noise. It is shown that the last model is characterized by bounded slopes, whereas in the EW model the slopes are unbounded. This fact makes the KPZ model equivalent to the self-organized interface depinning model of Buldyrev and Sneppen.

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I. INTRODUCTION

In this paper we discuss the scaling properties of rough interfaces in physical systems with quenched disorder. By "rough" we mean interfaces whose width W diverges with the linear scale of the system like $W \sim L^{\chi}$, where χ is known as the "roughening exponent." Examples are two-fluid flows in porous media [1,2], the invasion of coffee into a paper [3], or the motion of the interface between magnetic domains in a spin system with quenched randomness [4,5]. Recently it has become clear that the qualitative nature of the advancement of interfaces in systems with quenched disorder is different from the type of motion seen under the effect of an external random noise [3,6,7]. The difference is that the dynamics of the interface $h(\mathbf{x},t)$ is sensitive to a random term that can be modeled as $\eta(\mathbf{x},h(\mathbf{x},t))$, a term that depends on the position of the interface. This difference puts systems with quenched and external noise in different universality classes.

The theoretical activity in this subject has received a significant boost by the introduction of very simple models that describe the invasion of an interface between two phases into a medium with quenched disorder. Models that attracted attention were the models introduced in Refs. [3–6]. Broadly speaking, the models fall into two classes. One class deals with discrete space-time and introduces local rules for the advance of an interface. This class of models is very useful as a template for the introduction of the concepts needed for the discussion of the scaling properties of the growing interface. One of the points of this paper is that the same concepts are also appropriate for the discussion of another class of models. This second class is described by partial differential equations like

$$\partial_t h(\mathbf{x},t) = \mathcal{F}_{\rm nl}(\nabla h(\mathbf{x},t)) + \eta(x,h(\mathbf{x},t)) + F \ , \qquad (1.1)$$

where $\mathbf{x} \in \mathbb{R}^d$, $\mathcal{F}_{\rm nl}(\nabla h(x,t))$ is some (generally nonlinear) function of the spatial derivatives $\nabla h(\mathbf{x},t)$, and F is some force that pushes the interface $h(x,t)$. It turns out that even the linear model which is known as the Edwards-Wilkinson model [8],

$$\partial_t h(\mathbf{x},t) = \nabla^2 h(\mathbf{x},t) + \eta(x,h(\mathbf{x},t)) + F \ , \qquad (1.2)$$

has nontrival scaling properties due to the quenched noise [9–11].

The general feature of the models under study is that there exists an interplay between the degree of quenched disorder and the force acting on the interface. For a small force the interface gets pinned by the disorder. When the force is increased the interface usually moves for a while until it is pinned again. There exists a critical value of the force, say F_c, above which the interface moves with a finite velocity. Below the transition, but close to it, the correlation length diverges as $F - F_c$ tends to zero.

In a recent paper [7] we showed how a qualitative understanding of the effects of quenched randomness can be turned, under stated assumptions, into a complete scaling theory of one particular model [6] of a self-organized interface depinning. In this paper we return to a more critical analysis of the basic assumptions of that approach, and point out which of them have a wider applicability. In those models for which the assumptions are applicable one can achieve a rather complete scaling theory.

We begin this discussion in Sec. II, where the notion of associated processes is explained in a general context. We consider some delicate issues involving the definition and the measurement of roughness, and the relationship

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between the cluster roughness and the average properties of the interface. We propose that the roughness of such a system can depend on the details of the driving. In Sec. III the relation between the roughening of the growing interface and the scaling of the associated processes is discussed. We relate the scaling properties of the associated processes to other critical problems like the random walk and directed percolation that lead to alternative scaling relations for growth problems with quenched disorder. The alternative possibilities are tested against extensive numerical simulations, and conclusions are drawn. In Sec. IV we discuss continuous models and their relation to the discrete versions. As a part of this development we offer a scaling solution of the Kardar-Parisi-Zhang (KPZ) model [12] with quenched disorder and argue that this model is in the same universality class as the Buldyrev-Sneppen model. In addition we treat the Edwards-Wilkinson model and compare it to the KPZ model. We explain why these two models are in two different universality classes. In Sec. V we consider extensions to $2+1$ dimensions, including the roughening in the random-field Ising model. Section VI offers a summary and conclusions.

II. ROUGHENING AND ASSOCIATED PROCESSES

For concreteness we start with the discrete model introduced in Ref. [6]. In $1+1$ dimensions, the model is defined with a space coordinate x, $x=1,2,\ldots,L$, and with a "height" coordinate h, which is unbounded. Every lattice point (x,h) is assigned a quenched uncorrelated random number $f(x,h)$ in the interval [0,1]. A discrete interface $h(x)$ is grown from the initial condition $h=0$. The interface is updated in two steps. (i) The site is found with the smallest random number $f(x,h(x))$ among all the sites in front of the interface. On this site one unit is added to h, i.e., $h(x)\rightarrow h(x)+1$. (ii) After updating, the neighboring sites $y=x\pm1$ are checked for their local slope $|h(y)-h(y\pm1)|$. If this slope is greater than unity, the height $h(y)$ is adjusted by adding one unit. The same test is now applied to other sites, until $|h(z)-h(z\pm1)|\leq 1$ for all points z. The last step can result in an avalanche of adjustments, and is therefore referred to as such.

The roughness exponent of the interface χ was defined by the scaling of the "width" W,

$$W\equiv\sqrt{\langle[h(x,t)-\langle h\rangle]^2\rangle}\sim L^{\chi} \tag{2.1}$$

where $\langle\ \rangle$ denotes an average over space and members of the ensemble generated by different realizations of $f(x,h)$. Equation (2.1) is meant to hold for times t large compared to some saturation time $T_{\rm sat}$. Note that this definition can be extended to situations in which the interface is not a graph by averaging over the multivalued height functions. A number of groups measured the numerical value of χ and there appears to be a consensus that $\chi=0.63\ldots$. We review first the theoretical framework and then present some comments on this issue.

The connection to directed percolation (DP) was explained [3,13] as follows. Pick a value of $f(x,h)$ which is precisely $f_c=1-\rho_c$, where ρ_c is the critical density of the directed percolation problem. Consider now (see Fig. 1) the points (x,h) for which $f(x,h)\geq f_c$. Since f is uniform in the interval [0,1], the density of these points is ρ_c, and therefore they form a connected cluster (solid line in Fig. 1) which can have dangling bonds which are, however, directed. We denote this cluster by $C_{\rm DP}(f_c)$. Consider next the subset of this cluster, which consists of all the points that belong to the unique connected path with minimal values of $h(x)$ among all uninvaded sites, which we denote by $B_{\rm DP}(f_c)$ (see Fig. 1). The growing interface has to identify at some instant of time with a surface which for every x has a value of h which is smaller by one unit compared to $B_{\rm DP}(f_c)$ (points denoted by circles in Fig. 1). To see this, notice that any point x,h which is below this surface has a value of $f(x,h)$ lower than f_c, and will be chosen for growth before any point in $B_{\rm DP}(f_c)$. The only exceptions are the dangling bonds, which, however, are going to be circumvented and covered by avalanches. Thus the interface in Fig. 1 must identify with $B_{\rm DP}(f_c)$ before proceeding. In the limit $L\rightarrow\infty$ we can also state that the next point to be chosen will be a point for which $f=f_c$ *on* $B_{\rm DP}(f_c)$, and later this interface will be punctured by avalanches.

For any value of $f<f_c$ we can identify a directed percolation network $C_{\rm DP}(f)$ that has a finite density of points $\rho(f)$, which increases when f gets removed from f_c. Considering a situation that the highest value of the quenched noise in front of the interface is smaller than some f, we can define a blocking surface $B_{\rm DP}(f)$. This surface is defined as the lowest connected path belonging to $C_{\rm DP}(f)$. Evidently, the interface has to identify on its way with paths removed down one unit in h for each and every such cluster $B_{\rm DP}(f)$. It was suggested [3,13] that, although a wide range of values of f are selected during growth, the scaling of the interface will be dominated by the largest scales in the system, which are obtained when

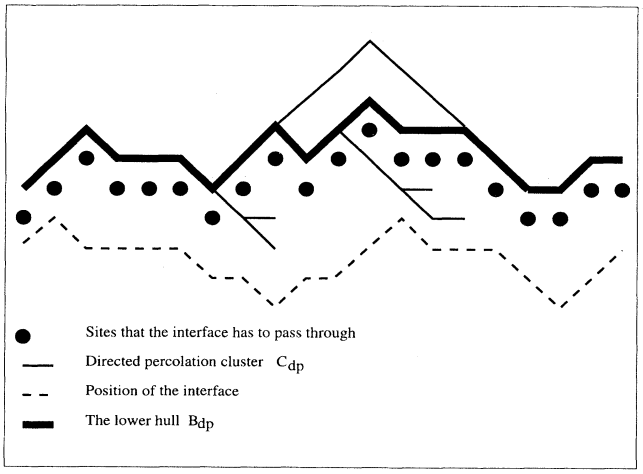

FIG. 1. A typical configuration of the growing interface with respect to the blocking directed percolation interface $C_{\rm DP}$ and the lower hull $B_{\rm DP}$. All solid lines pertain to $C_{\rm DP}$. The bold line is $B_{\rm DP}$. The interface has to identify with the path denoted by the dotted sites before it penetrates through $B_{\rm DP}$.

f is not too far from f_c. Then one can use the results for the transverse and parallel correlation lengths for the directed percolation problem [14], i.e.,

$$\xi_\perp \sim |\rho-\rho_c|^{-1.097} , \tag{2.2a}$$

$$\xi_\parallel \sim |\rho-\rho_c|^{-1.733} , \tag{2.2b}$$

to estimate a roughening exponent through

$$W \sim L^{1.097/1.733} = L^{0.633} . \tag{2.3}$$

This is in close agreement with the measured value of χ.

There are two crucial assumption that can go wrong in this argument. The first is that the critical exponents in Eqs. (2.2) really characterize the sizes of directed percolation clusters *below* the critical point, and not necessarily the roughening exponent of $B_{\rm DP}(f_c)$. The second is that the interface identifies with $B_{\rm DP}(f_c)$ only infrequently. Most of the time it identifies with noncritical paths $B_{\rm DP}(f)$. In principle this can give rise to changes in the scaling exponent even if the first problem is absent. This issue is discussed at some length in Sec. II B. Lastly, it is possible (and has also been demonstrated) that there exist correlations between events of growth separated in space and time, and these may contribute [15] to the roughening exponent in a way that needs to be understood. The discussion of these assumptions is facilitated by the introduction of the concept of associated processes.

A. The associated processes and their scaling properties

To be able to examine this issue further, we need to decompose the growth process into events that have cleaner scaling properties. These events are the "associated processes" which were introduced in Ref. [7], and which we recall now. Suppose that at time $t=t_0$ the point $x_0,h(x_0)$ was chosen for growth, and that $f(x_0,h(x_0))\geq f_0$. This implies that $f(x,h(x))\geq f_0$ on all the interface. Define the f_0 associated process, denoted as $A_{f_0}(s)$, as the series of steps in which the points $x_1,x_2,\ldots,x_s$ were chosen for growth, if the following conditions are met:

$$\text{(i)}\quad f(x_1)<f_0,\quad f(x_2)<f_0,\quad \ldots,\quad f(x_s)<f_0 , \tag{2.4a}$$

$$\text{(ii)}\quad f(x_{s+1})\geq f_0 . \tag{2.4b}$$

The following properties of the associated processes were demonstrated in Ref. [7].

(i) In the limit $L\to\infty$ there do not exist finite associated processes with $f_0>f_c$.

(ii) The set of points $x_1,x_2,\ldots,x_s$, together with the points exposed by avalanches, is simply connected and compact.

(iii) A consequence of (i) and (ii) is that, given a value of $f_0<f_c$, the union of the associated processes $A_{f_0}(s)$ is the lattice x,h. In other words, every value of f_0 defines a network in which the strings are given by the blocking surfaces characterized by f_0, and the holes in the net are the associated processes, with points $\{x_j\}_{j=1}^s$ plus avalanches. It was argued that this network is *not* the same as the network obtained by a directed percolation cluster of points with a density $\rho>\rho_c$.

(iv) Define $K_{f_0}(s)$ to be the distribution of s values in the union of f_0 associated processes. We assume that this distribution has the scaling form

$$K_{f_0}(s)=s^{-\tau}g(s/\Delta f^{-\nu}) , \tag{2.5}$$

where $\Delta f=f_c-f_0$, and τ and ν are scaling exponents.

The f_0 associated processes, being compact geometrical objects, afford a definition of measurable lengths $r_\parallel$ and width $r_\perp$. The definition is not unique. The associated process is bounded by two boundaries; by subtracting one boundary from another we get a new function $\tilde{h}(x)$. The length $r_\parallel$ is going to be defined as the length in which $\tilde{h}(x)$ is larger than zero. The length $r_\perp$ is defined as the width (2.1) of the function $\tilde{h}(x)$. One can define $r_\perp$ in a number of other ways, and crossover effects may lead to apparently different scaling exponents. We believe that for sufficiently large associated processes and sufficiently large systems these crossover effects disappear. The fundamental scaling exponent is obtained by measuring the conditional average width $\langle r_\perp|r_\parallel\rangle$ of all associated processes having a given length $r_\parallel$:

$$\langle r_\perp|r_\parallel\rangle \sim r_\parallel^{\chi_c} , \tag{2.6}$$

where χ_c is not necessarily identical to χ.

It is worthwhile to search for a measurement algorithm that will expose the existence of χ_c in this problem. One way to see this exponent is by examining the scaling properties which relate the conditional average area (number of growth steps A plus the number of avalanches) in the f_0 associated processes to the length $r_\parallel$ of the process:

$$\langle s|r_\parallel\rangle \sim \langle r_\perp|r_\parallel\rangle r_\parallel . \tag{2.7}$$

Note that (2.7) is a scaling assumption that needs to be checked. Using (2.6) in Eq. (2.7) one finds

$$\langle s|r_\parallel\rangle \sim r_\parallel^{1+\chi_c} . \tag{2.8}$$

We checked that (2.8) is reasonably supported by the data. The numerical value of the exponent in (2.8) is 1.62 ± 0.01.

In fact one can measure the roughening exponent in a surrogate growth process under different rules. In this process, rather than choosing the minimal value of f for the next growth one chooses *randomly* a site among all the growth sites whose $f\geq f^*$ with f^* close to f_c. Next one performs the usual algorithm of choosing the minimal f until the f^* associated process is completed. The width is then measured, and again a *random* site with $f\geq f^*$ is chosen, etc. We performed such a measurement with $f^*-f_c=5\times10^{-4}$ and $L=256\,000$ (the conditions are such that $\xi_\parallel\sim L$). The results (see Fig. 2) are in extremely good agreement with the directed percolation exponent $\chi=0.633\pm0.002=\chi_c$. This is a demonstration of the fact that the critical surfaces have the same statistics as typical directed percolation paths. In passing we note that direct roughness calculations of the directed percolation paths that we performed gave in-

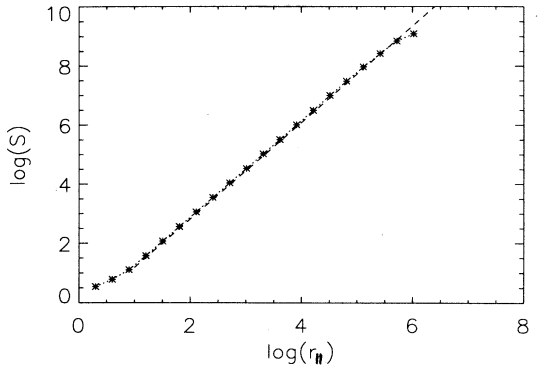

FIG. 2. The scaling of the area of a randomly chosen associated process with its length. The scaling exponent is observed to be 1.633. The stars represent the measured values and the dashed line represents scaling according to the exact scaling law. In this measurement the system size is $L=50\,000$. The scaling behavior spans four decades.

distinguishable scaling plots from those obtained in this surrogate process.

In Ref. [7] it was assumed that $\chi_c=\chi$. The rationale of this assumption was that associated processes with $f_0\to f_c$ must have a size comparable to L. It was thus felt that for L large enough the scaling properties of the growing interface were dominated by the scaling exponent of the largest associated processes, which are very close to criticality. One should stress here that if χ were larger than χ_c, different definitions of $r_\perp$ may lead to different evaluations of χ_c. Since this problem has not been carefully discussed in the literature we turn now to a more detailed discussion.

B. The interface roughness vs the roughness of the associated processes

In this subsection we will argue theoretically that χ_c is a lower bound on the value of χ. The main point of difficulty is that most of the measurements of the width are done after choosing an f value that can be rather far from f_c. One needs, therefore, to take into account noncritical contributions that may lead to a stable difference between χ and the directed percolation expected value of 0.633.

To establish a bound, consider now a finite system of size L and conditional sampling of the interface growth, where each sampling is done after the completion of an associated process with $f=f^*$, such that the length scale $\xi_\parallel(f^*)=2L$. Any associated process with $f_\parallel>L$ will generate an interface with roughness χ_c, since such a process wipes out any memory of the initial conditions. We will argue that there exists a finite fraction of associated processes at this value of f whose value of $r_\parallel$ is a small fraction of L, say αL, with α of the order of 0.1. Moreover, we will show that such processes contribute a finite fraction of samples of the width in which $\chi=\chi_c$. This will be the first element in the establishment of the bound.

It will be shown later that the exponent τ in Eq. (2.5) takes on values between 1 and 2. Therefore, the ratio R_s of the number of steps of the associated process whose $r_\parallel$ is smaller than αL to the total number of steps is proportional to

$$R_s\sim\frac{\int_1^{(\alpha L)^{1+\chi}} s^{-\tau}s\,ds}{\int_{(\alpha L)^{1+\chi}}^{L^{1+\chi}} s^{-\tau}s\,ds}\sim\alpha^{(1+\chi)(-\tau+2)}\,, \tag{2.9}$$

where we used the fact that the dominant contribution to the integrals comes from the upper limit. Since the smaller associated processes cannot affect the total roughness, it is clear that during all the steps included in them the measured width will scale with χ_c. The larger associated processes may well have a different roughness, say χ_1. Consider then the weighted measurement of the width between the two types of processes of sizes smaller and larger than αL, respectively. They will contribute to the width the weighted sum

$$W\sim R_s L^{\chi_c}+(1-R_s)L^{\chi_1}\,. \tag{2.10}$$

There are two possibilities: either $\chi_1\le\chi_c$ or $\chi_1>\chi_c$. In the first case χ_1 will be an irrelevant correction to scaling that will disappear in the limit $L\to\infty$. In the other case χ_1 will dominate the scaling and $\chi\ge\chi_c$. It is reasonable to assume that χ_1 will be larger since the growth for small f's is a very correlated process.

This argument indicates that the observed value of χ may depend on the chosen value of f^*, since the fraction R_s depends on f^*. To test this we measured for one value of L the dependence of the measured W on f^*, by sampling the measurement on the completion of f^* associated processes. The results are shown in Fig. 3. There is a systematic decrease in the observed values of W, which is a strong indication that χ_1 differs from χ_c. As a further test of these arguments we simulated this model in the range of L between 10^3 and 2.5×10^5. To avoid far from critical contributions we conditioned our measurements of W to times after the completion of an associated process with $f>f^*$, where $|f^*-f_c|=0.001\,76$. The results are shown in Fig. 4 in a double logarithmic plot. We show $W^2/L^{2\chi_c}$ as a function of L. Note that this measurement calls for an accurate knowledge of f_c. This

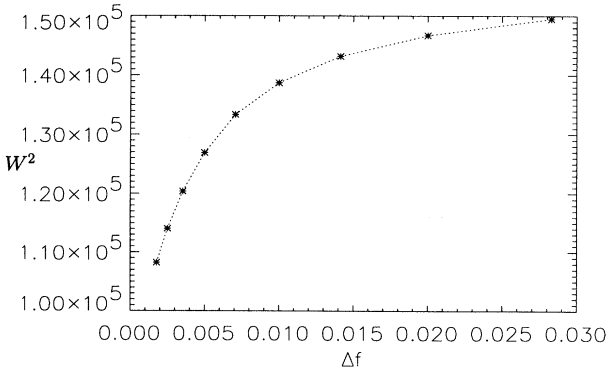

FIG. 3. The square of the width of the interface as a function of Δf. One can observe a decrease of the width with the decrease of f.

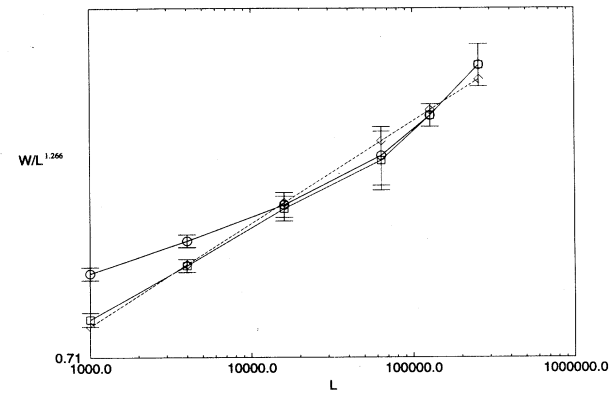

FIG. 4. The width of the interface as a function of the system size L for $\Delta f=0.002$. The graph is given in two versions, with and without a correction to scaling because of the shift in the critical value f_c with L. The circles represent values without shifts in f_c. Squares are data in which the shift is included. The diamonds represent the exact scaling law with a scaling exponent 1.32. One can observe that the effective scaling exponent is higher than the expected value of 1.266.

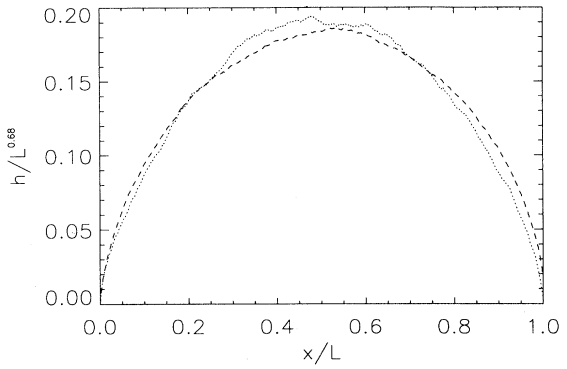

FIG. 5. The average profile of the interface in rescaled coordinates. The position $h(x)$ of the interface is scaled by $L^{0.68}$ and the coordinate by L, and many interfaces are averaged. The average interface is not flat, but has a width which scales with L with an exponent 0.68 ± 0.05. Shown are measurements for system sizes $L=16\,000$ and $128\,000$.

has been taken into account in Fig. 4 by estimating $f_c(L)$ from independent measurements. The exponent that is estimated from the results shown in Fig. 1 is $\chi=0.665\pm0.005$, which is significantly different from $\chi_c=0.633$. Though it is hard to estimate correctly because of measurement problems it seems that the exponent is even larger for larger systems.

We conclude that the roughness of the advancing surface is indeed larger than the roughness of the critical pinned surfaces. It is interesting to note that an experiment on advancing ink on an interface showed a different roughness for the pinned surfaces than for the advancing one ($\chi=0.7\pm0.04$). This is indicated by the previous discussion. *We believe that the details of the dynamics of the interface can be important.*

Lastly, we observe that measurements of the interface roughening exponent are influenced by another effect that to us seemed surprising. This has to do with the fact that the average over the growing interfaces when the boundary conditions are free is *not* flat. By a "free boundary condition" we mean that avalanches to positions smaller than $x=1$ and larger than $x=L$ are not performed, but the growth at $x=1$ and L is normal. Consider the quantity $h(x,t)-h(1,t)$. Average this quantity over time, $\langle h(x,t)-h(1,t)\rangle$. One could guess that this average would vanish. This is not the case, as can be seen in Fig. 5. The average width scales with system size with an exponent $\chi=0.68\pm0.05$. This indicates the existence of nonvanishing finite-size effects in this system which might modify the measurements of the roughness exponent from correlation functions; see Sec. V for a discussion of related problems. This effect might modify distributions and affect the statistics.

C. The effect of the rules of driving

The discussion in Sec. II B underlines the fact that noncritical effects may lead to roughening exponents that differ from the exponent of the critical interface. These noncritical effects may well depend on the rules of driving the interface. It is important therefore to understand the various rules of driving and their relation to physical procedures in experimental realizations of these processes.

In laboratory experiments one can choose constant current or constant driving. A constant current is realized by adjusting dynamically the driving force so that the pointwise volume invaded by the growing interface is fixed. In the previous section we discussed the case of constant current in the limit of infinitesimal current. In other words, only one point is activated per unit time. If we opted for constant driving, like constant pressure in the physical realization, many points would be invaded simultaneously. The constant infinitesimal current was realized in the previous section by the nonlocal search for an optimal site. This nonlocal search resulted in large fluctuations in the chosen value of the force f. These large fluctuations may result in stable deviations from the critical roughening exponent. On the other hand, if one chose a constant driving mechanism one would always be stopped by a critical surface when the force was exactly critical. In such a case χ should assume the precise value of χ_c. On the other hand, in the case of constant supercritical force one would find large fluctuations in the current even in close to critical conditions. These large fluctuations may lead to similar deviations in χ. These supercritical problems, including the finite correlation length that appears in such conditions, will be discussed in greater detail in the context of the continuous models below.

There exist additional rules that one can follow [3,5], and the aim of this subsection is to indicate that one needs to assess the details of the noncritical contributions.

III. SCALING RELATIONS

The basic scaling properties of the associated processes are given by the exponents in Eq. (2.5). We want to

derive scaling relations that connect the exponents ν, τ, and χ. In doing this we need to think about the nature of the network defined by the boundaries of the associated processes. It was explained in Ref. [7] that this network is *not* the same as the network of directed percolation which is obtained by eliminating all the dangling bonds from a directed percolation supercritical cluster. The basic reason for the difference stems from the rule of avalanches that eliminates potential boundaries in the network; see Fig. 2 of Ref. [7]. Accordingly, the scaling exponents that define the sizes of the associated processes may or may not be the same as the exponents that characterize the sizes of the holes of the directed percolation network. One way to think about the growing interface is as a random walk *on* a directed percolation network. The blocking surface $B_{\rm DP}(f)$ is a subset of the directed percolation network $C_{\rm DP}(f)$. At each node of $C_{\rm DP}(f)$ there is a random choice whether the interface continues upwards or downwards, depending on the values of f on the sites of $C_{\rm DP}(f)$. If we can indeed think about the interface as a random walk on the directed percolation network, this implies different scaling exponents for the associated processes. There are thus two alternative fundamental assumptions that lead to different scaling relations. One fundamental assumption is to simply state that ν in Eq. (2.5) is the same exponent as the one that determines the correlation length in the directed percolation problem [16]. This exponent is obtained by saying that

$$s=\xi_\perp\xi_\parallel\sim|\rho-\rho_c|^{(-1.097-1.733)} \quad \text{(directed percolation)} . \tag{3.1}$$

Another possibility to proceed stems from the fact that the boundaries of any given associated process are a Hurst random walk with the measured Hurst exponent χ_c. Accordingly, one can make a different fundamental assumption [7], i.e., that the distribution of $r_\parallel$ is identical to the distribution of intervals of return to the origin of a Hurst walk with exponent χ_c. This assumption yields different scaling relations from those implied by (3.1). We need to examine the two options, since the numerics does not appear to favor strongly either one. Both approaches share a number of scaling relations that are independent of the fundamental assumption. We derive these first.

A. General scaling relations

To derive the general relations we need two additional scaling exponents. Introduce the distributions of $r_\parallel$, denoted as $K_{f_0}(r_\parallel)$. We can write $K_{f_0}(r_\parallel)$, in correspondence with (2.5), in the scaling form

$$K_{f_0}(r_\parallel)=r_\parallel^{-\tau_\parallel}g\left[\frac{r_\parallel}{\Delta f^{-\nu_\parallel}}\right] . \tag{3.2}$$

In Ref. [7] it was shown that $\tau_\parallel$ and $\nu_\parallel$ are related:

$$1=\nu_\parallel(2-\tau_\parallel) . \tag{3.3}$$

In addition one has the trivial relationship

$$\nu=\nu_\parallel(1+\chi_c) , \tag{3.4}$$

which follows from $s=r_\parallel r_\perp\sim r_\parallel^{1+\chi_c}$. Another scaling relation that stems from this is

$$\tau=\frac{\tau_\parallel+\chi_c}{1+\chi_c} . \tag{3.5}$$

Last, we want to derive the average size of an associated process, which defines the exponent γ:

$$\langle s|f_0\rangle\sim\Delta f^{-\gamma} . \tag{3.6}$$

From (2.5)

$$\gamma=\nu(2-\tau) . \tag{3.7}$$

At this point the discussion bifurcates depending on the fundamental assumption. If one assumes that ν and $\nu_\parallel$ are determined by the directed percolation problem, then τ is determined from the knowledge of χ_c. On the other hand, if one assumes a knowledge of $\tau_\parallel$, one can determine ν. We describe now the two routes.

B. The two alternatives

Consider first the random-walk option. The probability of a Hurst random walk to return to the origin in an interval R scales like $R^{1-\chi_c}$. Accordingly, the mean interval size $\langle r_\parallel\rangle$ scales like $\langle r_\parallel\rangle$

$$\langle r_\parallel\rangle\sim R_{\chi_c} . \tag{3.8}$$

Using Eqs. (3.2) and (3.8) we see that

$$R^{\chi_c}\sim\int_D^R r_\parallel^{-\sigma_\parallel}r_\parallel dr_\parallel . \tag{3.9}$$

This determines $\tau_\parallel$ and γ as

$$\tau_\parallel=2-\chi_c,\quad \gamma=2 . \tag{3.10}$$

Thus if we accept that $\chi_c=0.633$ we expect from this approach that $\tau_\parallel=1.367$. Equations (3.3)–(3.5) furnish then numerical values for τ and ν.

Next consider the directed percolation option. In this case we use Eq. (3.3) to predict $\tau_\parallel=2-1/\nu_\parallel=1.423$. The difference in the numerical value of $\tau_\parallel$ will translate to different predictions for all the other exponents.

C. Measurements

Not having a way to choose theoretically between these two options, we attempted to distinguish between the predictions using numerical simulations. In our simulations we speeded up the calculations compared to the standard algorithm in order to achieve good statistics in large systems. The idea of the speedup is that instead of searching in every growth step for a global minimum of f we choose a certain $f_0<f_c$ and grow all the sites whose f are smaller than f_0 in parallel. This process generates f_0 associated processes. Although we lose the information about associated processes with lower values of f we speed up the simulations by factors of up to 1000.

As noted, the difference in $\tau_\parallel$ translates to differences in the other exponents. For example, the value of τ is

$2/(1+\chi_c)=1.225$ in the random-walk option but $\tau=1.259$ in the directed percolation option. In simulations (see Fig. 6) we got $\tau=1.245\pm0.003$. In addition we measured the average cluster size. This is easily done by measuring the frequency of finding different values of f. Measurements for a system of size 256 000 and 3×10^{11} growth events gave $\gamma=2.03\pm0.03$. The prediction for this exponent is $\gamma=2$ for the random-walk option and $\gamma=2.09$ for the directed percolation exponents. Consequently, the numerical results disagree in a similar manner with the predictions of the two approaches outlined above. It is possible that the strong finite-size effects discussed before (related to the existence of an average slope) affect the results in such a way that it is impossible to see clean scaling according to either approach.

As noted, the union of f_0 associated processes is not equivalent to a directed percolation network. Therefore we cannot assume that the exponent ν is given by the latter as was suggested in Refs. [13,16]. To calculate the exponent ν within the random-walk option we use Eq. (3.8) in Eqs. (3.3)–(3.5) and obtain

$$\nu=(1+\chi_c)/\chi_c \ . \tag{3.11}$$

This value of ν is lower than the corresponding directed percolation value. This fact poses a theoretical problem for the random-walk option. The size of the holes in the growth network of the union of the associated processes is *larger* than the size of the holes of the directed percolation cluster. It is thus impossible to have a value of ν for our network that is lower than the size ν for the holes of the directed percolation network. However, the value of ν in directed percolation is referred to the size of the percolation *clusters* and not of the holes in them. It is not known whether the latter scale like the former, and it is possible that the value of ν for the holes in the clusters is smaller than ν for the clusters themselves.

We tried to assess this issue numerically. Unfortunately it turned out that the measurement of ν for the holes is inconclusive. We cannot conclude on the basis of numerics which of the two options is true for the Buldyrev-Sneppen model, and it is possible that the final truth lies elsewhere. We will see that in the context of other models the random-walk option seems in excellent agreement with the data.

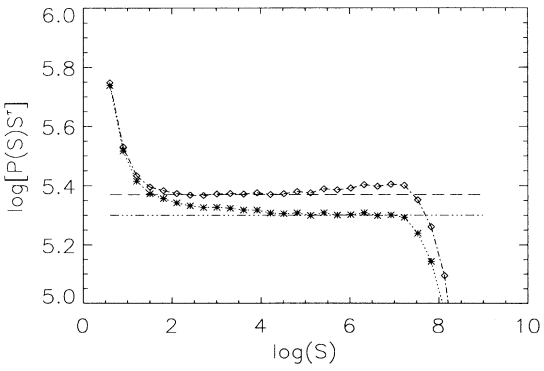

FIG. 6. A log-log plot of the distribution of areas of associated processes in the Buldyrev-Sneppen model. The distribution is rescaled by s^{τ} where $\tau=1.244$ (stars) and $\tau=1.259$ (diamonds). The former is the "best" fit as judged by flatness of the curve at large values of s, and the latter is the prediction of the directed percolation option. The horizontal lines are an aid to the eye to test for a flat region.

IV. MODELS IN CONTINUOUS SPACE AND TIME

In this section we show that much of the territory covered in Secs. II and III can be readily used to make progress in understanding the scaling properties of stochastic partial differential equations (PDE's) that describe interface roughening. There are again two types of dynamical laws that we analyze. The first type, which will be called "constant drive," is represented by equations of motion of the type

$$\partial_t h(x,t)=\mathcal{F}_{\rm nl}(\partial_x h(x,t))+\eta(x,h(x,t))+F \ , \tag{4.1}$$

where $\mathcal{F}_{\rm nl}(\partial_x h(x,t))$ is some (generally nonlinear) function of the spatial derivatives $\partial_x h(x,t)$, η represents the effect of the quenched disorder, and F is some force that pushes the interface $h(x,t)$. In our thinking η is chosen randomly from a reasonably bounded interval (say, uniform on [0,1]) and with some short spatial correlation length denoted as ξ_η. The second type of dynamical laws fixes the rate of advance of the interface $\partial_t h(x,t)$; the interface is moved at a point which maximizes (or minimizes) an internal field $F_{\rm int}$ that can be computed as

$$F_{\rm int}=\mathcal{F}_{\rm nl}(\partial_x h(x,t))+\eta(x,h(x,t)) \ .$$

We refer to this type of models as "constant current" models. Note that there exists an inherent difficulty in realizing such a process due to the continuity of space and of the values taken by $F_{\rm int}$. In practice all PDE's are simulated on the machine in discrete space, but our definition also calls for a discrete advance of the interface. In fact the PDE (4.1) should be considered as shorthand for the above lattice dynamics, and the continuum limit is not entirely well defined. We believe that as long as the discretization is made on length scales smaller than ξ_η the results are representative of the continuous process.

The two types of dynamical laws have common features that are very reminiscent of the discrete models. There are subtle differences that we expose first in the constant flux models.

(i) There exists a critical value of $F_{\rm int}$, denoted F_c, defined as the minimum value of $\mathcal{F}_{\rm nl}(\partial_x h(x,t))+\eta(x,h(x,t))$ which is found during growth. There exists paths in x-h space for which $F_{\rm int}(x,h(x))\le F_c$. The collection of these paths constitutes a network N_{F_c}. During growth the interface must overlap with subsets of N_{F_c}.

(ii) Every path can be characterized by the largest value of $F_{\rm int}$, say F^*, which is found on it. We shall refer to such a path as an F^* path. The collection of all the F^* paths forms a network denoted as C_{F^*}. During growth

an interface will identify with an F^* path. Next the growth process will cover a compact region of x-h space in which there is no point with $F_{\rm int}<F^*$ before the interface identifies again with an F^* path. These compact growth regions are termed "F^* associated processes." For $F^*>F_c$ we can consider all the paths for which $F_{\rm int}<F^*$, and these define the network N_{F^*}, of which C_{F^*} is necessarily a subset. If we denote by s the number of sites covered in an associated process, the distribution of s values in the union of F^* associated processes, $K_{F^*}(s)$, scales as in Eq. (2.5), i.e., $K_{F^*}(s)=s^{-\tau}g(s/\Delta F^{-\nu})$ where $\Delta F=F^*-F_c$.

(iii) The network C_{F_c}, which obtains for $F_{\rm int}=F_c$, is critical in the sense that it is characterized by two divergent correlation lengths $\xi_\perp$ and $\xi_\parallel$ which are the maximal length and width $r_\perp$ and $r_\parallel$ of the associated processes. For $F_{\rm int}$ close to F_c these correlation lengths are finite, and depend on $|F_{\rm int}-F_c|$ as $\xi_\perp\sim|F_{\rm int}-F_c|^{-\nu_\perp}$ and $\xi_\parallel\sim|F_{\rm int}-F_c|^{-\nu_\parallel}$. As we argued in Sec. II, the roughening exponent of the interface is bounded from below by the exponent that is obtained from the relation between $\xi_\perp$ and $\xi_\parallel$, which is

$$\xi_\perp\sim\xi_\parallel^{\nu_\perp/\nu_\parallel}=\xi_\parallel^{\chi_c} . \tag{4.2}$$

This relation defines the first scaling relation for the associated processes $\chi_c=\nu_\perp/\nu_\parallel$.

For the first type of model the language changes slightly, but the constructs of the theory remain. First, there exists an identical critical value of F, F_c, such that for every $F\le F_c$ there exist paths $h(x)$ for which $\partial_t h(x,t)=0$. For $F>F_c$ there are no such paths. These paths define a network N_F that depends on the value of F, which divides the x-h space into nonoverlapping regions. During the growth process, which is achieved by increasing F infinitesimally, the interface moves from one path in the subset N_F to a path in the subset N_{F+dF} by filling compactly regions of the x-h space. These regions are the analogs of the associated processes of the constant flux models, and they have the same scaling properties in $|F-F_c|$. The constant flux models organize themselves into a critical state, whereas in the first type of models the correlation length depends on F, and only at F_c is there a second order phase transition with divergent correlations.

The universal scaling relations in the continuous models do not change with respect to Eqs. (3.3)–(3.5). As before, one needs one additional relation between the scaling exponents in order to close the equations. In general, the continuous models do not necessarily belong to the directed percolation class, and we choose to follow here the random-walk alternative. There is one important difference from the discussion in Sec. III, which is the possibility that χ_c is larger than unity. [This is the situation, for example, in the Edwards-Wilkinson (EW) model as discussed below.] The difference appears first when we estimate the number of returns to the origin $N(R)$ in an interval of length R:

$$N(R)\sim\int_1^R r^{-\chi}dr\sim R^{1-\chi} \quad \text{if } \chi<1 , \tag{4.3}$$

$$N(R)\sim 1 \quad \text{if } \chi>1 . \tag{4.4}$$

The mean interval size $\langle r_\parallel|F\rangle$ scales like R^χ or like R for $\chi<1$ and for $\chi>1$, respectively. Using Eq. (3.8) we determine $\tau_\parallel$ as before, $\tau_\parallel=2-\chi$. For $\chi>1$ we find $\tau_\parallel=1$. We can thus write the general relation which pertains for any value of χ,

$$\tau=\max\left\{1,\frac{2}{1+\chi}\right\} . \tag{4.5}$$

The numerical value of χ is important also in the relation between ν and χ. Using Eqs. (3.2)–(3.4) together with the value of $\tau_\parallel$ found before we derive

$$\nu=\min\{(1+\chi),(1+\chi)/\chi\} . \tag{4.6}$$

A. Examples

To see the generality of this class, we consider now a few examples. The first is the Edwards-Wilkinson (EW) model [4,8]

$$\partial_t h(x,t)=\partial_x^2 h(x,t)+\eta(x,h(x,t))+F . \tag{4.7}$$

In discrete notation, defining $\Delta_i=h(x_i)-h(x_{i-1})$, the network C_F is obtained from the union of the paths $h(x_i)$ that satisfy

$$\Delta_i-\Delta_{i-1}=-\eta(x_i,h(x_i))-F , \tag{4.8}$$

together with the boundary conditions. Such a path blocks the growth with a given F: consider an interface that touches in at least one point, say x_j, a member of C_F. Denote the slopes of the path belonging to C_F by Δ_j^0 and Δ_{j+1}^0, and the slopes belonging to the interface by Δ_j and Δ_{j+1}. An inspection shows that $\Delta_j-\Delta_{j+1}+\eta(x_j,h(x_j))+F$ is necessarily negative, and therefore $h(x_j)$ cannot grow. Notice that the condition for stationarity implies that Δ_i can be unbounded. This is in contrast with the next model that we discuss below.

Numerical simulations of this model [11] in which the interface is advanced by running the system at $F=F_c$ have found a roughening exponent $\chi=1.25\pm0.02$. We have run this model according to the second type of algorithms searching the maximal value of $F_{\rm int}$, and found the same value of χ. The fact that $\chi>1$ is a sign that indeed the slopes are unbounded, and scale with L to a positive exponent which is larger than 0.25. Using the above results we predict that $\tau=1$, and $\nu=2.25\pm0.02$. From Eq. (3.7) we find $\gamma=2.25\pm0.02$. These predictions are easy to check numerically, and are confirmed in Fig. 7.

The same model has been analyzed theoretically in Ref. [10], with the prediction that in $1+1$ dimensions $\chi=1$ and $\nu_\parallel=1$. These predictions are not borne out by the simulations or by our scaling relations. In our notation one of the scaling relations of Ref. [10] is $\nu_\parallel=1/(2-\chi)$, which is in disagreement with our scaling relation $\nu_\parallel=\min\{1/\chi,1\}$ which in the present case is $\nu_\parallel=1$.

The next example is the KPZ model with quenched disorder [12],

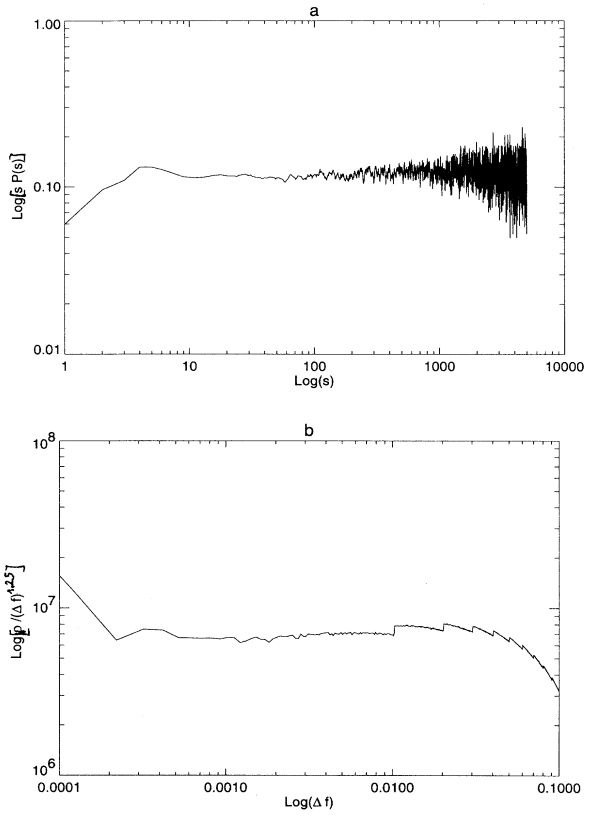

FIG. 7. (a) The size distribution of the associated processes for the EW model multiplied by the size. The flatness of the plot means that $\tau=1$. (b) A histogram of local field values selected at growth sites for the EW model divided by $|F-F_c|^{1.25}$. The flatness indicates that $\gamma=2.25$; see Ref. [7].

$$\partial_t h(x,t)=\partial_x^2 h(x,t)+\lambda|\partial_x h(x,t)|^2 +\eta(x,h(x,t))+F \ . \tag{4.9}$$

By running this model according to the second type of rules we discovered that this system belongs to the universality class of the Sneppen-Buldyrev model,

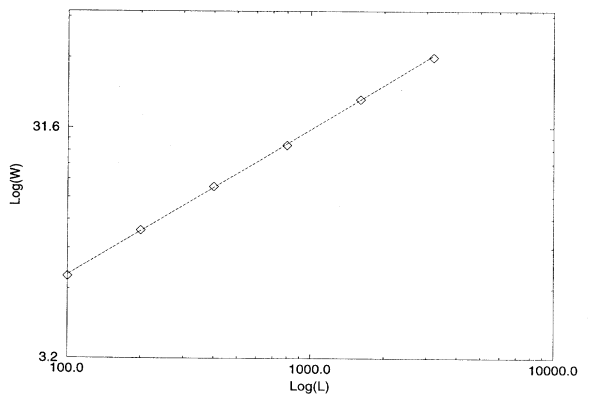

FIG. 8. A log-log plot of the interface width W vs the system size L in the KPZ model with quenched disorder. The line has a slope of 0.63.

$\chi\approx 0.63$, and the blocking surfaces are of the directed percolation type (see Fig. 8). To understand why this system is in a different universality class compared to Eq. (4.7) we can write again (4.9) in discrete notation, and find that the variables Δ_j are now bounded. The requirement for a blocking surface is

$$\lambda(\Delta_j+\Delta_{j+1})^2\le\Delta_j-\Delta_{j+1}+\eta_j+F \ .$$

Seeing this inequality as a quadratic inequality for Δ_j in terms of Δ_{j+1} (or vice versa), the condition for the existence of a real solution is that the discriminant will be positive. This is equivalent to the conditions

$$8\Delta_j\ge -1/\lambda+4\eta_j+4F, \quad 8\Delta_{j+1}\le 1/\lambda-4\eta_j-4F \tag{4.10}$$

at every site. Since η_j is bounded the blocking surfaces have bounded slopes, explaining why they could belong to the directed percolation universality class. In Fig. 8 we show a measurement of χ. It should be noted that (4.9) is in the same universality class as the Sneppen-Buldyrev model irrespective of the mode of driving, i.e., with a constant critical force or by selecting a minimally stable site.

Note that when $\lambda\to 0$ we should recapture the EW model (4.7) and indeed the slopes may become unbounded as (4.10) indicates. It is interesting to measure the mean slope in this model as a function of λ. Such a measurement is reported in Fig. 9, with the result that the mean slope depends on λ like $\lambda^{-\mu}$, with $\mu=0.26\pm 0.025$. This exponent is significantly smaller than the naive prediction of Eq. (4.10). To understand why the slope is considerably smaller we can average Eq. (4.9) over all the sites of a blocking interface. Denoting such an average by $\langle\ \rangle_c$ we write

$$\lambda\langle|\partial_x h(x,t)|^2\rangle=\langle\eta(x,h(x,t))+F\rangle_c \ . \tag{4.11}$$

The right-hand side (RHS) has a dependence on λ and it goes smoothly to zero for the critical surfaces of the EW model. One can see this immediately from Eq. (4.8) averaged over a blocking surface. We can therefore write

$$\langle\eta(x,h(x,t))+F\rangle_c\sim C\lambda^{\alpha} \tag{4.12}$$

where $\alpha>0$. Accordingly we get immediately a tighter bound

$$\langle|\partial_x h(x,t)|\rangle<\frac{1}{\sqrt{\lambda}} \ , \tag{4.13}$$

which is realized when $\alpha=0$. We did not succeed in deriving theoretically the exponent μ since we could not estimate α from first principles. The reason for the difficulty is that it appears that the two contributions on the LHS of (4.12) reach their EW limit with different leading exponents.

B. The effect of tilting the interface

In Ref. [17] simulations that pertain to the effect of introducing boundary conditions that produce an average tilt of the interface were presented. The mean slope is denoted in that work as m, and it is interesting to examine the results of these simulations in light of the general

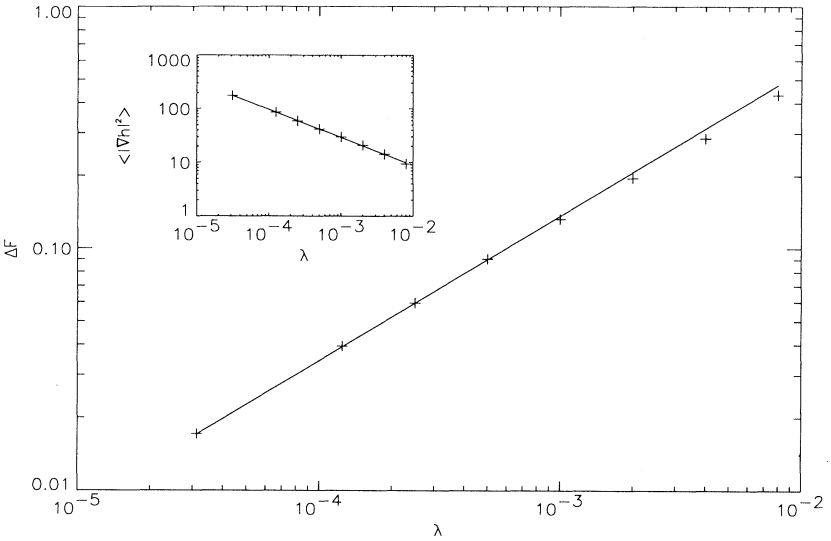

FIG. 9. The dependence of $\langle|\partial_x h(x,t)|^2\rangle$ and the critical force ΔF on λ for the KPZ model with quenched noise for a system of size $L=4000$. The measured exponents are given in the text.

approach developed here (and see also Ref. [18]).

The existence of an average tilt influences the properties of the network of the associated processes. Remember that without a tilt every hole in the network has two scales $r_\perp$ and $r_\parallel$ that characterize its height and width, respectively. The parameter m should be compared for each hole in the network to the ratio $r_\perp/r_\parallel$, which from Eq. (2.6) scales like $r_\parallel^{\chi-1}$. This comparison introduces a new typical scale which is obtained from finding that slope, say $(r_\perp/r_\parallel)^*$, which equals the external tilt m:

$$(r_\perp/r_\parallel)^*=m \ . \tag{4.14}$$

The value of $r_\parallel$ for which (4.14) is satisfied is the scale $\xi^*(m)$ which is

$$\xi^*(m)=m^{1/(\chi-1)} \ . \tag{4.15}$$

Obviously no holes of length larger than $\xi^*(m)$ may exist in such an experiment. This means that the threshold of depinning is reduced compared to the untilted case. The amount of reduction ΔF^* is estimated from equating the correlation length of the untilted network with $\xi^*(m)$. Using $\xi_\parallel \sim \Delta F^{-\nu_\parallel}$ we get from equating the scales

$$\Delta F^* \sim m^{1/\nu_\parallel(\chi-1)} \ . \tag{4.16}$$

The prediction is then that the exponent appearing in (4.16) is about 1.57. We tested this prediction numerically and found satisfactory agreement.

These considerations allow also an evaluation of the mean velocity v of propagation of the interface for the constant drive models at the critical point $F=F_c$. The way to think about it is as follows. Define as the fundamental unit of time the time Δt that is required to fill a hole of size $\xi_\perp\xi_\parallel$. In order to advance the interface by an amount $\xi_\perp$ we need to fill $L/\xi_\parallel$ individual holes. However, all these holes are being filled simultaneously in the constant drive algorithm. Accordingly, we advance the whole interface by $\xi_\perp$ during a time that is inversely proportional to $\xi_\parallel$. Accordingly we can estimate

$$v \sim C\xi_\perp/\xi_\parallel \sim \Delta F^{\nu_\parallel-\nu_\perp} \equiv \Delta F^\theta \ . \tag{4.17}$$

This result was obtained first in Ref. [13] using similar arguments. We can proceed now to use it to estimate the dependence of the velocity on the imposed tilt m.

At the critical point there is one typical scale, i.e., $\xi^*(m)$. Using (4.17) with ΔF read from (4.16) we obtain

$$v \sim m \ , \tag{4.18}$$

where we have used $\nu_\perp=\nu_\parallel\chi$. This prediction was checked and the good agreement is shown in Fig. 10.

The prediction of Eq. (4.18) pertains to the critical condition $\Delta F=0$. In supercritical conditions $\Delta F>0$ the dependence on m is nonlinear [17]. To understand this phenomenon we generalize (4.17) to the finite-tilt situation by introducing the scaling function $g(x)$ through the relation

$$v=\Delta F^\theta g(m/\Delta F^\theta) \ . \tag{4.19}$$

The properties of $g(x)$ are that

$$g(x)\sim \begin{cases} 1, & x\to 0 \\ x, & x\to\infty \end{cases} \ . \tag{4.20}$$

The symmetry of the problem to the transformation $m\to -m$ requires that $g(x)$ will attain a minimum at $x=0$. If we assume that $g(x)$ is smooth in x we predict that the x dependence for small x is generically quadratic (which is not obvious from first principles). We thus expect that for supercritical conditions

$$v \sim \Delta F^\theta + \Delta F^{-\theta} m^2 \ . \tag{4.21}$$

The simulations of Ref. [17] seem to be in accord with this dependence. We note that in Ref. [17] it was proposed that the quadratic dependence on m as exhibited in (4.21) is a result of the quadratic term $|\partial_x h(x,t)|^2$ in the

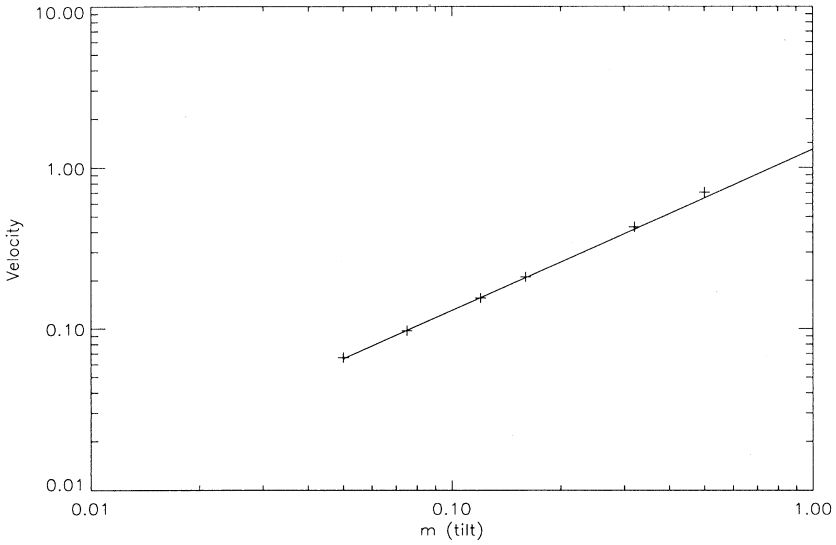

FIG. 10. The linear dependence of the velocity on the tilt m, shown for system size $L=100\,000$. Because of finite-size effects there is no current below the lowest point.

equation of motion. Our argument shows that it is simply a consequence of reflection symmetry, and that it cannot in general be tied to the quadratic term in the equation. It should also be added at this point that one should not expect such a quadratic m dependence in the EW model simply because small tilts are irrelevant in that case in which $\chi=1.25$. In a model in which $\chi>1$ small tilts are irrelevant perturbations and their effect is negligible.

Finally we note that the growth process in a tilted situation gains new features on scales larger than $\xi^*(m)$. On such large scales the connection to the directed percolation problem is totally destroyed, and the growth occurs as a series of uncorrelated downward avalanches with trivial dynamical exponents. One expects that on such a large scale the roughness will cross over to the random-walk universality class of $\chi=\frac{1}{2}$.

V. MODELS IN 2+1 DIMENSIONS

In $2+1$ or in higher dimensions, one exponent is surely not sufficient to define all the scaling properties. The random-walk argument cannot be used. The difficulty is in the argument that led to Eq. (3.10). That equation is a reflection of the fact that when we cut in $1+1$ dimensions a random path of fractal dimension $1+\chi$ we get a Cantor set of points of dimension χ. This can be used to estimate the number of points in an interval and therefore of the mean distance between the points as executed above. In contrast, in $2+1$ dimensions we have a surface of dimension $2+\chi$, but when we cut it we get islands whose *perimeter* may have a dimension of $1+\chi$, but whose *area* cannot be deduced from the dimension of the mother surface. Even worse, generically the distribution of areas of islands in cuts is not characterized by a single power law. There is usually a large continent among the islands, and the area of the continent may be dominant in the distribution. The subject of random cuts of random surfaces is a rather involved one, and we cannot do justice to it in the present paper.

Fortunately, our simulations indicate that *the distribution of associated processes is critical.* A distribution

$$K_{F^*}(V)=V^{-\tau}g(V/\Delta f^{-\nu}) \tag{5.1}$$

is available, where V is now the volume of the F^* associated process. As before we define an index γ by $\langle V\rangle_{F^*}\sim\Delta f^{-\gamma}$. The full characterization of the growth process in terms of geometry and dynamics calls now for two independent exponents, say χ and τ. In making this statement we make the nontrivial assumption that the associated processes are still compact. This property is supported by simulations in $2+1$ dimensions, but may well be lost in higher dimensions.

Since the set C_{F^*} is made now of surfaces rather than lines, every associated process is characterized by an area A and height $r_\perp$. We have now $V\sim Ar_\perp$ and $r_\perp\sim A^{\chi/2}$. Therefore

$$K_{F^*}(A)=A^{-\tau_A}g(A/\Delta f^{-\nu_A}), \tag{5.2}$$

where $\tau_A=\tau(2+\chi)/2+\chi/2$ and $\nu_A=2\nu/(2+\chi)$. (Note that if we want to define a length $r_\parallel$ we can use $r_\parallel=\sqrt{A}$ which defines $\nu_\parallel=\nu_A/2$.) From considerations identical to those leading to Eqs. (3.3) and (3.7) we find

$$1=\nu_A(2-\tau_A),\quad \gamma=\nu(2-\tau)\ . \tag{5.3}$$

A. Examples

The first example that we discuss is the Buldyrev-Sneppen model [19] in $2+1$ dimensions. The exponents χ and τ were measured in Ref. [19] and by us. The results are $\chi=0.51\pm0.03$, and $\tau=1.46\pm0.02$. Using Eq. (5.3) we can predict $\gamma\approx1.53$ and $\nu_\parallel\approx1.15$. These predictions were confirmed by simulations in Ref. [19] and by us. Note that in Ref. [19] the measured result for $\nu_\parallel$ is in close agreement with these numbers, whereas the result for γ seems unreasonably small, $\gamma=1.41$.

Next we discuss the random-field Ising model (RFIM)

[20]. The dynamics of this model differs from Eq. (4.1). The Hamiltonian is

$$H=\sum_{\mathbf{r}\mathbf{r}'} JS_{\mathbf{r}}S_{\mathbf{r}'}+\sum_{\mathbf{r}}\eta(\mathbf{r})S_{\mathbf{r}}+H\sum_{\mathbf{r}}S_{\mathbf{r}}\ , \tag{5.4}$$

and $\mathbf{r}$ is a coordinate in three-space that we denote as $(\mathbf{x},z)$. The sum on $\mathbf{r},\mathbf{r}'$ is taken on nearest neighbors, and $\eta(\mathbf{r})$ is a random local field taken from a bound distribution of width Δ. One begins with initial conditions with $H=-\infty$, and the row of spins at $z=0$ is chosen with $S_{\mathbf{r}}=1$, whereas all the other spins are -1. The value of the field H is raised now quasistatically. A spin $S_{\mathbf{r}}$ is flipped to 1 if the local field

$$H_L\equiv\sum_{\mathbf{r}} JS_{\mathbf{r}}+\eta(\mathbf{r})+H \tag{5.5}$$

exceeds zero, and if $\mathbf{r}$ is a nearest neighbor site to the interface. It is known [20] that for $J/\Delta\ll 1$ the interface assumes the geometry of an invasion percolation cluster, whereas for $J/\Delta\gg 1$ it remains flat. There exist intermediate values of J/Δ for which the interface becomes self-affine.

In Ref. [20] the roughening exponent χ was measured through the scaling of the standard deviation of the interface $h(x)$ as a function of the coordinate x for a given total length L of the system. We remark that although the interface is self-affine it is *not a graph*. The interface has overhangs which ruin the possibility of estimating χ from such measurements. To understand the difficulty, imagine that we want to extract the scaling exponent from the measurement of correlation functions. The existence of overhangs leads to a wrong estimate of the roughening exponent when measured from correlation functions. As an example consider an "interface" in the shape of a unit step function at $x=0$. Computing the correlation $\langle[h(x+r)-h(x)]^2\rangle$ we get a contribution $h(x+r)-h(x)=0$ whenever x and $x+r$ are on the same side of the step, and a contribution $h(x+r)-h(x)=1$ whenever x and $x+r$ are on the two sides of the step. Therefore $\langle[h(x+r)-h(x)]^2\rangle\sim r$. This is an example of a wrong identification of $\chi=\frac{1}{2}$ that would result from steps. Overhangs are smoothed in interface calculations as steps, and therefore correlations cannot be used to estimate χ. A similar problem will appear when the standard deviation as a function of x is measured, since the scale of the overhangs will contaminate the scaling behavior. The scale of the overhangs as a function of system size changes with an exponent that in general is nonzero and different from χ.

A practical way to avoid being misled is to compute higher order structure functions. Overhangs and steps may lead to fake multiscaling. In the previous example all the higher order structure functions $\langle[h(x+r)-h(x)]^n\rangle$ will exhibit the same fake scaling exponent $\chi=\frac{1}{2}$ instead of an exponent that is linear in n. The sign of trouble will be observed in the inner cutoff of the scaling behavior. The size of the typical overhang (or step) scales with the size of the system. Since the small scale behavior is dominated by the steps, the smallest observed scale in the scaling regime will grow with the system size. If, on the contrary, a measurement of higher order structure functions results in proper scaling exponents which are simple multiples of the first order exponent, overhangs are irrelevant. We remark that the EW model with quenched disorder is another example where the correlation function method will fail to measure the correct exponent $\chi=1.25$, for similar reasons. Since in that case the slopes diverge with L with an exponent, say ψ, the correlations display an apparent exponent $(\chi-\psi)$ instead of the true roughness exponent.

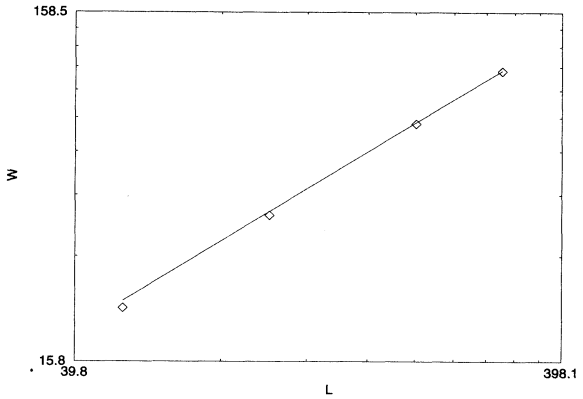

FIG. 11. A log-log plot of the interface width W vs system size L for the random-field Ising model in 2+1 dimensions. The line has a slope of 0.85.

We measured χ from the width of the interface $W\sim L^{\chi}$, and for $J/\Delta=2.8$. We found $\chi=0.85\pm0.02$; see Fig. 11. The scale of the overhangs changes like $L^{0.2}$. This rationalized the measurement in Ref. [5(a)] of an apparent exponent of $\frac{2}{3}$. The exponents γ, τ_v, and ν were properly measured in Ref. [5] and we repeated their measurements. Reference [5] found $\tau_v=1.28\pm0.05$, $\gamma=1.71\pm0.1$, and $\nu_{\perp}=0.75\pm0.05$. The scaling relations that we presented fitted the data very well, since we predict $\gamma=1.71$ and $\nu_{\perp}=0.72$. We remark that the scaling relations offered in Ref. [20] agree with ours only when $\chi=1$, which is not the usual case. In fact, Refs. [5] and [20] assumed that the scaling of the interface differs from the scaling of the associated processes, and that the roughness of the interface is *smaller* than the roughness of the associated processes. As we explained in Sec. II B this is not possible.

We have measured numerically the scaling of the associated processes in this model and found that they have a χ exponent which is very close to the exponent of the interface. This is another strong indication that the relevant roughness definition in this model is inherited from the width of the interface and not from the properties of the correlation functions.

VI. SUMMARY

In summary, we showed that the scaling relations that were derived in Ref. [7] have a much wider applicability than anticipated before. It appears that one can offer a unified framework to discuss the scaling properties of moving interfaces in a wide variety of models with

quenched disorder. On the other hand, the model in which the scaling relations were originally derived (the Buldyrev-Sneppen model) shows stubborn deviations from the predictions of the scaling relations, independently of the choice of the random-walk or the direction percolation options. The deviations are small, but it is possible that they indicate something deep. It is also possible however that the model is plagued with corrections to scaling as explained in Secs. II and III.

In all the other models discussed above the scaling relations seem to agree very well with the results of numerical simulations.

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